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Major Goals: Systems modeled by a large number of dynamic interacting particles have long been of interest in Statistical Physics. In recent years similar models have started appearing in many other fields as well. These include, communication systems (e.g. loss network models, random medium access protocols), mathematical finance (e.g. mean field games, default clustering in large portfolios), chemical and biological systems(e.g. biological aggregation, chemotactic response dynamics), neuroscience and social sciences (e.g. opinion dynamics models.) The objective of this project is to develop mathematical theory that enables to predict the behavior of the system when the number of particles is very large, with reliable error bounds, particularly when the system is in steady state. The mathematical results that we are interested in take the form of
• Law of large numbers and Central Limit Theorems.
• Concentration inequalities for probabilities of deviations from nominal behavior, Large and Moderate deviation principles.
• Optimal control and dynamic mean field games.
• Time asymptotic behavior of the system and numerical schemes with provable uniform in time convergence properties.
In our work we consider such problems for many different models arising from broad range of applications. The mathematical tools for our work come from the theory of stochastic dynamical systems, Markov processes, Stochastic PDE, stochastic stability and control and general stochastic process theory. The motivating applications come from communication systems, neurosciences, biological models, financial mathematics etc. Two specific applications we focus on are (i) active biological transport and (ii) models for neuronal assembly. Under (i), as a prototype we consider a system that has been proposed for chemotaxis of cells that actively modify the chemical field through their aggregated input with cell dynamics governed by the gradient of the same chemical field. We are interested in developing a general asymptotic theory that covers not only this model but also many other variations that have been introduced in literature to study phenomenon of trunk trail formation, biological switches, self organized networks, search algorithms etc. For (ii) the basic questions of interest are as follows. The brain consists of a large number of neurons whose firing rates are characterized by complex interacting nonlinear dynamics. Noise, both intrinsic and extrinsic, forms an important part of the neuronal signal and plays a fundamental role in brain function. The firing statistics of an individual neuron are highly stochastic yet reliable and predictable responses to specific stimuli arise as a result of collective dynamical behavior of large ensembles of such interconnected neurons that share a common input. Understanding the effect of single neuron variability and dependence structure between neurons on the dynamics of large cortical networks is one of the major challenges of modern neuroscience. We will consider models for such neural networks given in terms of weakly interacting diffusion processes. Our goal is the development of a rigorous asymptotic theory for such processes and identification of useful reduced systems that can be used for qualitative insights and quantitative predictions. Such

reduced models can also be extremely useful for approximate simulation of network dynamics.

Accomplishments: Research conducted under this grant was primarily focused on three main areas: (A) Weakly Interacting Stochastic Particle Systems, (B) Large Deviation Behavior of Stochastic Systems, (C) Control of Stochastic Networks.
Topics covered in this research include mean field models, multiscale systems, large stochastic networks with mean field interaction, dynamics on large inhomogeneous random graphs, fluctuation results, large deviation principles, optimal stochastic control and stability of nonlinear Markov processes.
In all, the work completed under this grant resulted in publication/acceptance of 16 papers and 7 additional papers are currently under review.
One Ph.D. student (Abhishek Pal Majumder) completed his dissertation under the joint supervision of the PI and Prof. Hannig in May 2015 and a second Ph.D. student (Ruoyu Wu) completed dissertation in
May 2016. Two other Ph.D. students (Eric Friedlander and Yang Yu) continued work on their dissertations. Three postdoctoral fellows (Louis Fan, Dane Johnson, Michael Perlmutter) were partially supported by the grant as well.
In the attachment we describe our new results in the three main topics listed above and also give summaries of the dissertation works of the two Ph.D. students partially supported by the grant. Invited presentations given by the PI during the period of the grant are also listed.

Training Opportunities: One Ph.D. student (Abhishek Pal Majumder) completed his dissertation under the joint supervision of the PI and Prof. Hannig in May 2015 and a second Ph.D. student (Ruoyu Wu) completed dissertation in May 2016. Two other Ph.D. students (Eric Friedlander and Yang Yu) are continuing work on their dissertations.
These four students have been partially supported by the grant and their dissertation is on some topics proposed in the grant proposal.
Two postdoctoral fellows (Louis Fan and Dane Johnson) were partially supported by the grant as well. Joint papers with postdoctoral fellows on topics in the grant are being prepared.
PI also developed and taught a graduate course in Spring 2016 on Large Deviation topics closely related to the grant proposal.
Postdoctoral fellow Louis Fan mentored and supervised six undergraduate students on four research projects (students: Emily Riederer, David Clancy
Clay Hackney, Lili Chen, Samir D. Patel, Dongzhi Zheng, David Qiu Hongxiang.)
Posters based on these projects were presented at `The Celebration of Undergraduate Research (CUR)' -- an annual research symposium for UNC undergrads.

Results Dissemination: PI gave four invited presentations at major conferences in 2014-2016 on topics studied under this grant. PI also disseminated his research activities through talks at departmental colloquia at six universities during this period.
The attached PDF document has details on these presentations and talks.
PI's students and postdocs presented two posters at the Seminars in Stochastic Processes 2016 (U. Maryland), one poster at the Stochastic Networks Conference 2016 (U. Cal. San Diego), one half hour invited talk at the 4th IMS Asia Pacific Rim Meeting 2016 (U. of Hong Kong) and one half hour contributed talk at the World Congress in Probability and Statistics 2016 (Field Institute).

Honors and Awards: 1. Invited to join as the permanent member of the Scientific Committee of the Seminar on Stochastic Processes.
2. Member of the Program Committee for the 4th Institute of Mathematical Statistics Asia Pacific Rim Meeting.

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Theory and Applications of Weakly Interacting Markov Processes. Final Report.

Agreement number W911NF-14-1-0331, Proposal number 65602MA
Amarjit Budhiraja
Department of Statistics and Operations Research
University of North Carolina at Chapel Hill

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Research conducted under this grant was primarily focused on three main areas: (A) Weakly Interacting Stochastic Particle Systems, (B) Large Deviation Behavior of Stochastic Systems, (C) Control of Stochastic Networks. Topics covered in this research include mean field models, multiscale systems, large stochastic networks with mean field interaction, dynamics on large inhomogeneous random graphs, fluctuation results, large deviation principles, optimal stochastic control and stability of nonlinear Markov processes. In all, the work completed under this grant resulted in publication/acceptance of 16 papers and 7 additional papers are currently under review. One Ph.D. student (Abhishek Pal Majumder) completed his dissertation under the joint supervision of the PI and Prof. Hannig in May 2015 and a second Ph.D. student (Ruoyu Wu) completed dissertation in May 2016. Two other Ph.D. students (Eric Friedlander and Yang Yu) continued work on their dissertations. Three postdoctoral fellows (Louis Fan, Dane Johnson, Michael Perlmutter) were partially supported by the grant as well. Below we describe our new results in the three main topics listed above and also give summaries of the dissertation works of the two Ph.D. students partially supported by the grant. Invited presentations given by the PI during the period of the grant are also listed.

1 Research on Weakly Interacting Stochastic Particle Systems

Weakly interacting particle systems have long been of interest in Statistical Physics (see [76, 60] and references therein). In recent years similar models have started appearing in many other fields as well. These include, communication systems (e.g. loss network models[50, 2, 77], random medium access protocols[54, 68]), mathematical finance (e.g. mean field games[65, 55, 37, 59], default clustering in large portfolios[43, 44, 51]), chemical and biological systems(e.g. biological aggregation, chemotactic response dynamics[73, 75, 49, 20]), social sciences (e.g. opinion dynamics models [39, 53]). A typical such model describes a N-dimensional Markov process $X^N = (X_1^N, \cdots X_N^N)$ with sample paths in $C([0,\infty) : \mathcal{S}^N)$ (space of continuous functions from $[0,\infty)$ to $\mathcal{S}^N$ equipped with the local uniform topology where $\mathcal{S}$ is some Polish space). The transition probabilities of this Markov process are such that if the initial distribution, namely the probability law of $X^N(0)$, is exchangeable then X^N is exchangeable is as well. The interaction between the particles takes place through the empirical distribution of the N-particles and consequently the influence of a typical particle on the evolution of any other particle os $O(1/N)$. For this reason this interaction is referred to as a weak interaction.

In our work we have studied several basic mathematical properties of such weakly interacting systems, including law of large numbers (LLN) and central limit theorems (CLT), asymptotic probabilities of deviations from the mean behavior, long time behavior of the system, stochastic control and stochastic differential games associated with mean field systems. We now provide additional details on these works.

1.i Law of large numbers and Central limit theorems.

Law of large numbers and central limit theorems provide tractable approximate models for complex weakly interacting systems. In our work we have studied several such systems.

(1.i.a) Central Limit Results for Jump-Diffusions with Mean Field Interaction and a Common Factor [28]. In this work we consider a system of N weakly interacting particles whose dynamics is given in terms of jump-diffusions with a common factor. The common factor is described through another jump-diffusion and the coefficients of the evolution equation for each particle depend, in addition to its own state value, on the empirical measure of the states of the N particles and the common factor. Systems with a common factor arise in many different areas. In Mathematical Finance, they have been used to model correlations between default probabilities of multiple firms. In neuroscience modeling these arise as systematic noise in the external current input to a neuronal ensemble. For particle approximation schemes for stochastic partial differential equations (SPDE), the common factor corresponds to the underlying driving noise in the SPDE. The goal of this work is to study a general family of weakly interacting jump-diffusions with a common factor. Our main objective is to establish a suitable Central Limit Theorem(CLT). A key point here is that due to the presence of the common factor, the limit of particle empirical measures will in general be a random measure. This in particular means that the centering in the fluctuation theorem will typically be random as well and one expects the limit law for such fluctuations to be not Gaussian but rather a 'Gaussian mixture'. Our main result provides a CLT under quite general conditions. The summands in this CLT can be quite general functionals of the trajectories of the particles with suitable integrability properties. The key idea is to first consider a closely related collection of N stochastic processes that, conditionally on a common factor, are independent and identically distributed. By introducing a suitable Radon-Nikodym derivative one can evaluate the expectations associated with a perturbed form of the original scaled and centered sum in terms of the conditionally i.i.d. collection. The asymptotics of the latter quantity are easier to analyze using, in particular, the classical limit theorems for symmetric statistics. The perturbation arises due to the fact that in the original system the evolution of the common factor jump-diffusion depends on the empirical measure of the states of the N-particles whereas in the conditionally i.i.d. construction the common factor evolution is determined by the large particle limit of the empirical measures. Estimating the error introduced by this perturbation is one of the key technical challenges in the proof. An application to models in Mathematical Finance of self-excited correlated defaults is described.

(1.i.b) Some Fluctuation Results for Weakly Interacting Multi-type Particle Systems [35]. In this work we extend the results of [28] to multi-type populations. We consider a collection of N-diffusing interacting particles where each particle belongs to one of K different populations. Evolution equation for a particle from population k depends on the K empirical measures of particle states corresponding to the various populations and the form of this dependence may change from one population to another. In addition, the drift coefficients in the particle evolution equations may depend on a factor that is common to all particles and which is described through the solution of a stochastic differential equation coupled, through the empirical measures, with the N-particle dynamics. We are interested in the asymptotic behavior as $N \to \infty$. Such multi-type systems have been proposed as models in social sciences [40], statistical mechanics [38], neurosciences [5], etc.

Although for multi-type settings the full system is not exchangeable, particles in the same population have an exchangeable distribution. Using this structure, one can prove using standard techniques a law of large numbers result and a propagation of chaos property[5]. In this work we study fluctuations about the law of large number limit. For the case where the common factor is absent the limit is given in terms of a Gaussian field whereas in the presence of a common factor it is characterized through a mixture of Gaussian distributions. We also obtain, as a corollary, new fluctuation results for disjoint sub-families of single type particle systems, i.e. when $K = 1$. Finally, we establish limit theorems for multi-type statistics of such weakly interacting particles, given in terms of multiple Wiener integrals.

(1.i.c) Weakly Interacting Particle Systems on Inhomogeneous Random Graphs [15]. The systems considered in (a) and (b) above correspond to settings where each particle interacts with every other particle (albeit weakly). However, in many network settings, interaction between particles is governed by a (possibly random) graph in that only the neighbors in the graph interact directly. In this work we consider weakly interacting diffusions on time varying random graphs. The system consists of a large number of nodes in which the state of each node is governed by a diffusion process that is influenced by the neighboring nodes. The collection of neighbors of a given node changes dynamically over time and is determined through a time evolving random graph process. Thus the model is described in terms of two types of stochastic dynamical systems, one that describes the evolution of the graph that governs the interaction between nodes of the system (*dynamics of the network*) and the other that describes the evolution of the states of all the nodes in the system (*dynamics on the network*). We consider a setting where the interaction between the nodes is *weak* in that the 'strength' of the interaction between a node and its neighbor is inversely proportional to the total number of neighbors of that node. Such stochastic systems arise in many different areas, such as social networks (e.g. in the study of gossip algorithms [10, 74]), biological systems (e.g. swarming and flocking models, see [11, 12] and references therein), neurosciences (e.g. in modeling of networks of spiking neurons, see [5, 17] and references therein), and mathematical finance (e.g. in modeling correlations between default probabilities of multiple firms [42]). In our work we establish a law of large numbers and a propagation of chaos result for a multi-type population setting where at each instant the interaction between nodes is given by an inhomogeneous random graph which may change over time. This result covers the setting in which the edge probabilities between any two nodes is allowed to decay to 0 as the size of the system grows. We also establish a central limit theorem for the single-type population case under stronger conditions on the edge probability function.

(1.i.d) Diffusion Approximations for Load Balancing Mechanisms in Cloud Storage Systems [26]. In the world of cloud-based computing, large data centers are often used for file storage. In these data centers, large networks of servers are used to store even larger sets of files. In order to improve reliability and retrieval speed, these files are often “coded”. By coded, we mean that the file is broken down into smaller pieces which are stored on multiple servers. Consider the situation in which there are four servers and one file. One can store the entire file on one server but in such a configuration the file would be inaccessible if that server were to fail. In order to improve reliability, one can replicate the file across all four servers but such a method would require much more memory. Suppose we instead split the file into halves, A and B, and then store $A,\ B,\ A+B,\ A-B$ in each of the four servers, respectively. Then the original file can be constructed from any two pieces. One can extend this idea to the case where equally sized pieces of a file are stored across L servers and any k pieces can reconstruct the original file. This can be accomplished using the Maximum Distance Separable (MDS) code with parameters (L, k) [67]. The MDS code greatly improves reliability since $L-k+1$ servers must fail before the file becomes irretrievable, while only requiring enough total memory to store L/k files. Given a coding scheme, one can consider load balancing mechanisms to improve file retrieval speed. In [66], two routing schemes, called Batch Sampling (BS) and Redundant Request with Killing (RRK), are considered. In BS routing, incoming jobs are routed to the k shortest queues containing the file being requested, while in RRK routing jobs are routed to all servers containing the requested file and then removed from the queue (killed) once k pieces of the file have been returned. The paper [66] formally calculates the steady state $(T \to \infty)$ queue length distribution in the large system limit $(n \to \infty)$ and gives simulation results for different values of L and k in both routing schemes.

In this work we are interested in developing a rigorous limit theory for such load balancing schemes for systems with MDS coding as n becomes large.We establish a law of large numbers and a central limit theorem as the system becomes large (i.e. $n \to \infty$). For the central limit theorem, the limit process take values in ℓ_2, the space of square summable sequences, and is described through an infinite dimensional stochastic differential equation driven by a cylindrical Brownian motion. Due to the large size of such systems, a direct analysis of the n-server system is frequently intractable. The law of large numbers and diffusion approximations established in this work provide practical tools with which to perform such analysis. The Power-of-d routing scheme, also known as the supermarket model, is a special case of the model considered here.

(1.i.e) Supermarket Model on Graphs [33]. In this paper we analyze a variation of the supermarket model in which the servers can communicate with their neighbors and where the neighborhood relationships are described in terms of a suitable graph. Specifically, consider a graph G_N on N vertices, where the vertices represent single-server queues. Tasks with unit-exponential service time distributions arrive at each server as independent Poisson processes of rate λ, and each task is irrevocably assigned to the shortest queue among the one it first appears and its $d-1$ randomly selected neighbors.

The above model has been extensively investigated in the case where G_N is a clique. In that case, each task is assigned to the shortest queue among $d \geq 2$ queues selected randomly from the entire system, which is commonly referred to as the 'power-of-d' or JSQ(d) scheme. Since the servers are exchangeable when the underlying graph is a clique, the system is quite tractable via classical mean-field techniques. Results in Mitzenmacher [70, 71] and Vvedenskaya *et al.* [79] show that for any fixed value of d, as the size of the clique gets large, the occupancy process associated with the queue-lengths at the various servers converges to a deterministic limit described by an infinite system of ordinary differential equations (ODE). Moreover, even sampling as few as $d = 2$ servers yields significant performance enhancements over purely random assignment ($d = 1$) as $N \to \infty$. Specifically, when $\lambda < 1$, the probability that there are i or more tasks at a given queue in steady state is proportional to $\lambda^{\frac{d^i-1}{d-1}}$ as $N \to \infty$, and thus exhibits super-exponential decay in λ as opposed to exponential decay for the random assignment policy.

Unfortunately, however, large-scale service systems often suffer from stringent locality constraints, and when a task arrives at any specific server, it becomes difficult, if not impossible, to fetch instantaneous state information from arbitrarily selected $d-1$ servers. Moreover, executing a task commonly involves the use of some data, and storing such data for all possible tasks on all servers will typically require an excessive amount of storage capacity [81, 80]. The above issues motivate consideration of sparser graph topologies where tasks that arrive at a specific server i are only allowed to be forwarded to a subset of the servers that can be thought of as neighbors in some graph G_N. While considering load balancing schemes with sparse topologies is desirable from applications perspectives, the corresponding mathematical formulation, that results in systems that in general will not be exchangeable or have simple Markovian state descriptors, puts us outside the range of classical mean-field techniques, leading to a fairly uncharted territory from methodological standpoint.

We show that if the minimum degree approaches infinity (however slowly) as the number of servers N approaches infinity, and the ratio between the maximum degree and the minimum degree in each connected component approaches 1 uniformly, the occupancy process converges to the same system of ODE as the classical supermarket model. In particular, the asymptotic behavior of the occupancy process is insensitive to the precise network topology. We also study the case where the graph sequence is random, with the N-th graph given as an Erdős-Rényi random graph on N vertices with average degree $c(N)$. Annealed convergence of the occupancy process to the same deterministic limit is established under the condition $c(N) \to \infty$, and under a stronger condition $c(N)/\ln N \to \infty$, convergence (in probability) is shown for almost every realization of the random graph.

1.ii Long time behavior of weakly interacting particle systems.

Limit theorems studied in items (i.a)-(i.e) concern the behavior of the system over a finite time horizon. In many applications the time asymptotic behavior of the system is of central concern. For example, stability of a communication system, steady state aggregation and self organization in biological and chemical systems, long term consensus formation mechanisms in opinion dynamics modeling, particle based approximation methods for invariant measures all rely on a careful analysis of the time asymptotic behavior of such systems. In our work we have studied such long time behavior for several families of weakly interacting particle systems. We now describe these results.

(1.ii.a) Long Time Results for a Weakly Interacting Particle System in Discrete Time [32]. In this work we consider a discrete time weakly interacting particle system and the corresponding nonlinear Markov process in $\mathbb{R}^d$, described in terms of a general stochastic evolution equation. Denoting by $X_n^i \equiv X_n^{i,N}$ the state of the i-th particle ($i = 1, ..., N$) at time instant n, the evolution is given as

$$X_{n+1}^i \quad = \quad AX_n^i + \delta f(X_n^i, \mu_n^N, \epsilon_{n+1}^i), \qquad\qquad i = 1, .., N, \quad n \in \mathbb{N}_0 \tag{1}$$

Here $\mu_n^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_n^i}$ is the empirical measure of the particle values at time instant n, A is a $d \times d$ matrix, δ is a small parameter, $\{\epsilon_n^i, i = 1, ..., N, \quad n \geq 1\}$ is an i.i.d array of $\mathbb{R}^m$ valued random variables with common probability law θ and $f : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^m \to \mathbb{R}^d$ is a measurable function. Also, $\{X_0^i, i = 1, ..., N\}$ are taken to be exchangeable with common distribution μ_0. The following nonlinear Markov chain corresponds to the $N \to \infty$ limit of (1).

$$X_{n+1} = AX_n + \delta f(X_n, \mu_n, \epsilon_{n+1}), \quad \mathcal{L}(X_n) = \mu_n, \ n \in \mathbb{N}_0. \tag{2}$$

where $\mathcal{L}(X)$ denotes the probability distribution of a random variable X with values in some Polish space S. Under conditions on model coefficients and parameters we study several long time properties of the N-particle system and the associated nonlinear Markov chain. Our starting point is the evolution equation for the law of the nonlinear Markov chain given by the equation

$$\mu_{n+1} = \Psi(\mu_n). \tag{3}$$

where $\Psi : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is defined as $\Psi(\mu) = \mu P^\mu$. We show that under conditions, that include a Lipschitz property of f with the Wasserstein-1($\mathcal{W}_1$) distance on the space of probability measures, contractivity of A and δ being sufficiently small, (3) has a unique fixed point and starting from an arbitrary initial condition convergence to the fixed point occurs at an exponential rate. Using this result we next argue that under an additional integrability condition, as $N \to \infty$, the empirical measure μ_n^N of the N-particles at time n converges to the law μ_n of the nonlinear Markov process at time n, in the $\mathcal{W}_1$ distance, in L^1, *uniformly* in n. This result in particular shows that the $\mathcal{W}_1$ distance between μ_n^N and the unique fixed point μ_∞ of (3) converges to zero as $n \to \infty$ and $N \to \infty$ in any order. This result is key in developing particle based numerical schemes for approximating the fixed point of the evolution equation (3). We next show that under an irreducibility condition on the underlying Markovian dynamics the unique invariant measure Π_∞^N of the N-particle dynamics is μ_∞-chaotic, namely as $N \to \infty$, the projection of Π_∞^N on the first k-coordinates converges to $\mu_\infty^{\otimes k}$ for every $k \geq 1$. This propagation of chaos property all the way to $n = \infty$ crucially relies on the uniform in time convergence of μ_n^N to μ_∞. Our results also give rates for this uniform convergence through polynomial and exponential concentration estimates.

(1.ii.b) Limits of relative entropies associated with weakly interacting particle systems [21]. The purpose of this work is to introduce and develop a systematic approach to the construction of Lyapunov functions for studying local stability properties of nonlinear Markov processes. We consider a collection of N weakly interacting particles, in which each particle evolves as a continuous time pure jump càdlàg stochastic process taking values in a finite state space $\mathcal{X} = \{1, \ldots, d\}$. The evolution of this collection of particles is described by an N-dimensional time-homogeneous Markov process $\boldsymbol{X}^N = \{X^{i,N}\}_{i=1,\ldots,N}$, where for $t \geq 0$, $X^{i,N}(t)$ represents the state of the ith particle at time t. The jump intensity of any given particle depends on the configuration of other particles only through the empirical measure

$$\mu^N(t) \doteq \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{i,N}(t)}, \quad t \in [0, \infty), \tag{4}$$

where δ_a is the Dirac measure at a. Such mean field weakly interacting processes arise in a variety of applications ranging from physics and biology to social networks and telecommunications, and have been studied in many works. It can be shown that on any fixed time interval $[0, T]$, the particles become asymptotically independent as $N \to \infty$, and that for each fixed t the distribution of a typical particle converges to a probability measure $p(t)$, which coincides with the limit in probability of the sequence of empirical measures $\{\mu^N(t)\}_{N \in \mathbb{N}}$ as $N \to \infty$. Under suitable conditions, the function $t \mapsto p(t)$ can be characterized as the unique solution of a nonlinear differential equation on $\mathcal{P}(\mathcal{X})$ of the form

$$\frac{d}{dt} p(t) = p(t)\Gamma(p(t)), \tag{5}$$

where for each $p \in \mathcal{P}(\mathcal{X})$, $\Gamma(p)$ is a rate matrix for a Markov chain on $\mathcal{X}$. This differential equation admits an interpretation as the forward equation of a "nonlinear" jump Markov process that represents the evolution of the typical particle.

The approach we take in constructing Lyapunov functions for such nonlinear Markov processes is to lift the problem to the level of the pre-limit N-particle processes that describe a linear Markov process. Under mild conditions the N-particle process will be ergodic, and thus relative entropy can be used to define a Lyapunov function for the joint distribution of these N particles. The scaling properties of relative entropy and convergence properties of the weakly interacting system then suggest that the limit of suitably normalized relative entropies for the N-particle system, assuming it exists, is a natural candidate Lyapunov function for the corresponding nonlinear Markov process. The main results of the paper prove convergence of such scaled relative entropies in various settings.

(1.ii.c) Local stability of Kolmogorov forward equations for finite state nonlinear

Markov processes [22]. This paper is a continuation of the work in (ii.b). The focus here is to use the Lyapunov functions constructed in the above work for the study of local stability of a class of nonlinear ordinary differential equations (ODE) that describe limits of empirical measures associated with finite-state exchangeable weakly interacting N-particle systems, described above. Local Lyapunov functions are identified for several classes of such ODE, including those associated with systems with slow adaptation and Gibbs systems. Using results from [21] and large deviations heuristics, a partial differential equation (PDE) associated with the nonlinear ODE is introduced and it is shown that positive definite subsolutions of this PDE serve as local Lyapunov functions for the ODE. This PDE characterization is used to construct explicit Lyapunov functions for a broad class of models called locally Gibbs systems. This class of models is significantly larger than the family of Gibbs systems and several examples of such systems are presented, including models with nearest neighbor jumps and models with simultaneous jumps that arise in applications.

(1.ii.d) Uniform in Time Interacting Particle Approximations for Nonlinear Equations of Patlak-Keller-Segel type [23]. In this work we study a system of interacting diffusions that models chemotaxis of biological cells or microorganisms (referred to as particles) in a chemical field that is dynamically modified through the collective contributions from the particles. Such systems of reinforced diffusions have been widely studied and their hydrodynamic limits that are nonlinear non-local partial differential equations are usually referred to as Patlak-Keller-Segel (PKS) equations. Solutions of the classical PKS equation may blow up in finite time and much of the PDE literature has been focused on understanding this blow-up phenomenon. In this work we study a modified form of the PKS equation that is natural for applications and for which global existence and uniqueness of solutions are easily seen to hold. Our focus here is instead on the study of the long time behavior through certain interacting particle systems. Under the so-called "quasi-stationary hypothesis" on the chemical field, the limit PDE reduces to a parabolic-elliptic system that is closely related to granular media equations whose time asymptotic properties have been extensively studied probabilistically through certain Lyapunov functions. The modified PKS equation studied in the current work is a parabolic-parabolic system for which analogous Lyapunov function constructions are not available. A key challenge in the analysis is that the associated interacting particle system is not a Markov process as the interaction term depends on the whole history of the empirical measure. We establish, under suitable conditions, uniform in time convergence of the empirical measure of particle states to the solution of the PDE. We also provide uniform in time exponential concentration bounds for rate of the above convergence under additional integrability conditions. Finally, we introduce an Euler discretization scheme for the simulation of the interacting particle system and give error bounds that show that the scheme converges uniformly in time and in the size of the particle system as the discretization parameter approaches zero.

1.iii Deviations of Weakly-Interacting Particle Systems from its Mean Behavior.

Probabilities of deviations from the mean behavior of a stochastic system are usually given my establishing a large or moderate deviation principle. In our work work we have studied such principles for several types of weakly interacting systems.

(1.iii.a) Moderate Deviation Principles for Weakly Interacting Particle Systems [36]. In this work moderate deviation principles for empirical measure processes associated with weakly interacting Markov processes are established. Two families of models are considered: the first corresponds to a system of interacting diffusions whereas the second describes a collection of pure jump Markov processes with a countable state space. For both cases the moderate deviation principle is formulated in terms of a large deviation principle (LDP), with an appropriate speed function, for suitably centered and normalized empirical measure processes. For the first family of models the LDP is established in the path space of an appropriate Schwartz distribution space whereas for the second family the LDP is proved in the space of l_2 (the Hilbert space of square summable sequences)-valued paths. Proofs rely on certain variational representations for exponential functionals of Brownian motions and Poisson random measures.

(1.iii.b) Large Deviations for Brownian Particle Systems with Killing [24]. Particle approximations for certain nonlinear and nonlocal reaction-diffusion equations are studied using a system of Brownian motions with killing. The system is described by a collection of i.i.d. Brownian particles where each particle is killed independently at a rate determined by the empirical sub-probability measure of the states of the particles alive. More precisely, let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. exponential random variables with rate 1 and let $\{B_i(t), t \geq 0\}_{i\geq 1}$ be independent d-dimensional standard Brownian motions independent of $\{X_i\}_{i\geq 1}$. Define for $t \geq 0$ the random sub-probability measure $\mu^n(t)$ as the solution to the following equation

$$\mu^n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{B_i(t)} 1_{\left\{X_i > \int_0^t \langle \zeta, \mu^n(s)\rangle \, ds\right\}}. \tag{6}$$

Since a.s., we can enumerate $\{X_i\}_{i=1}^n$ in a strictly increasing order, the unique solution of (6) can be written explicitly in a recursive manner. It can be checked that $\mu^n \doteq \{\mu^n(t)\}_{t\in[0,T]}$ converges, in the Skorokhod path space $\mathcal{D}$, in probability to μ where for $t > 0$, $\mu(t)$ has density $u(t, \cdot)$ given as the solution of a nonlinear PDE. Such particle systems are motivated by problems in biology, ecology, chemical kinetics, etc. For example, the simplest case where $\zeta \equiv 1$ corresponds to the case where the killing rate is proportional to the total number of particles alive and models a setting in which particles compete for a common resource. More general functions ζ are of interest as well and one interpretation of $\zeta(x)$ is the amount of resource consumed by a particle in state x (see for instance [48] and Chapter 9 of [78]). Similar particle systems arise in problems of mathematical finance as models for self exciting correlated defaults [42].

In our work a large deviation principle (LDP) for such sub-probability measure-valued processes is established. Along the way a convenient variational representation, which is of independent interest, for expectations of nonnegative functionals of Brownian motions together with an i.i.d. sequence of random variables is established. Proof of the LDP relies on this variational representation and weak convergence arguments.

1.iv Stochastic Control and Mean Field Games.

Weakly interacting particle systems also arise in many engineering applications where one is interested in optimal control of the system or the system is governed by strategic actions of a large number of competing players. In our work we have studied several such problems.

(1.iv.a) Diffusion Approximations for Controlled Weakly Interacting Large Finite State Systems with Simultaneous Jumps [25]. We consider a rate control problem for an N-particle weakly interacting finite state Markov process. The process models the state evolution of a large collection of particles and allows for multiple particles to change state simultaneously. Such models have been proposed for large communication systems (e.g. ad hoc wireless networks) but are also suitable for other settings such as chemical-reaction networks. An associated diffusion control problem is presented and we show that the value function of the N-particle controlled system converges to the value function of the limit diffusion control problem as $N \to \infty$. The diffusion coefficient in the limit model is typically degenerate, however under suitable conditions there is an equivalent formulation in terms of a controlled diffusion with a uniformly non-degenerate diffusion coefficient. Using this equivalence, we show that near optimal continuous feedback controls exist for the diffusion control problem. We then construct near asymptotically optimal control policies for the N-particle system based on such continuous feedback controls. Some numerical experiments are explored as well.

(1.iv.b) Rate Control under Heavy Traffic with Strategic Servers [6]. Rate controlled queueing systems commonly arise from applications in communication systems, see e.g. [46, 61, 3, 19] and references therein. Recently, they have also been considered in modeling limit order books, see e.g. [8, 9, 56, 18, 16, 63]. A common approach to the study of such rate control problems when the system is in heavy traffic is through diffusion approximations. In a problem setting where there is interaction between servers/queues in that the rates or costs associated with a particular queue and server can depend on the states of the other queues, this approach leads to a stochastic control problem for n-dimensional reflected diffusions, where n is the number of queues in the system. When n is large such control problems are computationally intractable and in general this 'curse of dimensionality' is unavoidable. However, when there are certain symmetries present and the interaction between queues is weak, in that each queue has $\mathcal{O}(1/n)$ affect on any other queue in the system, a natural approach is to consider, in addition to heavy traffic, another asymptotic regime where the number of queues n approaches ∞ as well. Such model settings arise in many applications, e.g., cloud computing, live streaming, limit order books, customer service systems, etc. In many of these contexts the servers are strategic, for example, in customer service networks, servers respond to workload incentives (see [52]), and in the context of limit order books buyers and sellers place their orders in a strategic manner and interact weakly through their impact on the price distribution.

In our work we consider a large queueing system that consists of many strategic servers that are weakly interacting. Each server processes jobs from its unique critically loaded buffer and controls the rate of arrivals and departures associated with its queue to minimize its expected cost. The rates and the cost functions in addition to depending on the control action, can depend, in a symmetric fashion, on the size of the individual queue and the empirical measure of the states of all queues in the system. In order to determine an approximate Nash equilibrium for this finite player game we construct a Lasry-Lions type mean-field game (MFG) for certain reflected diffusions that governs the limiting behavior. Under conditions, we establish the convergence of the Nash-equilibrium value for the finite size queuing system to the value of the MFG.

(1.iv.c) A numerical scheme for a mean field game in some queueing systems based on Markov chain approximation method [7]. In general closed form solutions for MFGs of the form considered above are not available and thus one needs numerical approximations. . In our work, we use the Markov chain approximation method to construct approximations for the solution of the mean field game (MFG) with reflecting barriers studied in [6]. The MFG is formulated in terms of a controlled reflected diffusion with a cost function that depends on the reflection terms in addition to the standard variables: state, control, and the mean field term. By showing that our scheme is an almost contraction, we establish the convergence of this numerical scheme over a small time interval. Our method, in contrast to previous numerical approaches, is purely probabilistic. We do not make smoothness assumptions that are usually made in numerical studies, instead we use an iterative Markov chain approximation method (see [62]) to construct numerical solutions of the MFG. Specifically, we discretize time and space and for a fixed measure on the path space we define a Markov decision problem that is suggested by the MFG. In the first step of the iteration, the law of the solution of the MDP is computed. Then we take this law as the starting point to formulate the MDP for the second iteration and repeat the process. Unfortunately, it is not clear that the map defined by such iterations is in general a contraction. We instead show that the map is an *almost contraction* over a small time interval with length independent of the discretization parameter. By an almost contraction we roughly mean that the map is a contraction up to an additional term that vanishes as the discretization parameter approaches 0. The proof of this almost contraction property relies on the construction of a coupling between certain controlled reflected Markov chains which we believe is of independent interest. Using the above almost contraction property, tightness of relevant processes, and weak convergence arguments, we show the convergence of the laws obtained from the iteration scheme to the solution of the MFG over a small time interval. Proving the convergence of a Markov chain based approximation method of the form considered in this work over an arbitrary time interval is for now a challenging open problem.

2 Research on Large Deviation Behavior of Stochastic Systems

Theory of large deviations is a well established field whose basic mathematical foundations were laid in the classical works of Donsker and Varadhan. The topic has been extensively studied and now there are several excellent books and monographs on the topic. The central goal in this theory is the study of precise asymptotics of probabilities of rare events associated with a given stochastic system as a certain scaling parameter approaches its limit. Consider for example the simplest such setting where we are given an iid real sequence of integrable random variables $\{X_k\}_{k\in\mathbb{N}}$. The large deviations question in this setting is to characterize the precise decay rate of $P(|S_n/n - \mu| \geq \varepsilon)$ as $n \to \infty$ for $\varepsilon > 0$, where $S_n = \sum_{k=1}^n X_k$ and $\mu = EX_1$. Under appropriate integrability conditions Cramér's theorem answers this question by giving a *large deviation principle* (LDP) that provides matching upper and lower bounds on the exponential decay rate in terms of the associated *rate function*. This basic problem of characterization of decay rates of probabilities of deviations from the nominal behavior of a stochastic system arises in a large number of problems and is a central theme in probability theory. Such problems go beyond the core area of probability theory and applications of the theory abound in all the physical and biological sciences and also in engineering, computer science, economics, finance, and social sciences. Large deviations theory not only gives asymptotic bounds for probabilities but also techniques for constructing efficient accelerated Monte-Carlo methods for estimating probabilities of rare events (see [45] and references therein). In our works we have studied large and moderate deviation asymptotics for several families of stochastic systems. A description of these results is as follows.

(2.i) Large deviations for multidimensional state-dependent shot noise processes [34]. Shot noise processes are used in applied probability to model a variety of physical systems in, for example, teletraffic theory, insurance and risk theory and in the engineering sciences. In this work we prove a large deviation principle for the sample- paths of a general class of multidimensional state-dependent Poisson shot noise processes. Such processes are natural models for systems where the impact of a shot depends on the current state value of the system. The result covers previously known large deviation results for one dimensional state-independent shot noise processes with light tails. In order to prove large deviation results we use the fact that a Poisson shot noise process can be represented as an integral with respect to a Poisson random measure. Using such representations, our work builds on certain variational formulas for functionals of a Poisson random measure and their application to large deviations. Rather than traditional large deviation techniques, we use the weak convergence approach to large deviations. In the current context this amounts to proving the appropriate convergence of certain controlled versions of the original process, together with the necessary existence and uniqueness results. The main advantage of the weak convergence approach is that it avoids the discretization/approximation arguments and exponential estimates typically encountered in a large deviation analysis. Such approximation methods are in general difficult to implement for complex settings such as the state-dependent shot noise processes considered here.

(2.ii) Large Deviations for Small Noise Diffusions in a Fast Markovian Environment [30]. We study a stochastic system with two time scales where the slow scale evolution is described through a continuous stochastic process, given by a small noise finite dimensional Itô stochastic differential equation, and the fast component is given as a rapidly oscillating pure jump process. The two processes are fully coupled in that the drift and diffusion coefficient of the slow process and the jump intensity function and jump distribution of the fast process depend on the states of both components. Multiscale systems of the form considered in this work arise in many problems from systems biology, financial engineering, queuing systems, etc. For example, most cellular processes are inherently multiscale in nature with reactions occurring at varying speeds. This is especially true in many genetic networks, where protein concentration, usually modeled by a small-noise diffusion process, is controlled by different genes rapidly switching between their respective active and inactive states [41]. The key characterizing feature of such slow-fast systems is that the fast component reaches its equilibrium state at much shorter time scales at which the slow system effectively remains unchanged. This local equilibration phenomenon allows the approximation of the properties of the slow system by averaging out the coefficients over the local stationary distributions of the fast component. Such approximations yield a significant model simplification and are mathematically justified by establishing an appropriate *averaging principle*.

In our work a large deviation principle is established for such two-scale stochastic systems that characterizes asymptotics of probabilities of deviations from the averaging principle. Previous works have considered settings where the coupling between the components is weak in a certain sense. In the current work we study a fully coupled system in which the drift and diffusion coefficient of the slow component and the jump intensity function and jump distribution of the fast process depend on the states of both components. In addition, the diffusion can be degenerate. Our proofs use certain stochastic control representations for expectations of exponential functionals of finite dimensional Brownian motions and Poisson random measures together with weak convergence arguments. A key challenge is in the proof of the large deviation lower bound where, due to the interplay between the degeneracy of the diffusion and the full dependence of the coefficients on the two components, the associated local rate function has poor regularity properties.

(2.iii) Large Deviation Principle for the Exploration Process of the Configuration Model [13]. The goal of this work is to study large deviation properties of certain random graph models. The system of interest is the so-called *configuration model* which refers to a sequence of random graphs with number of vertices approaching infinity and the degree distribution converging to a pre-specified probability distribution on the set of non-negative integers. The configuration model is a basic object in probabilistic combinatorics (cf. [58] and references therein) and is one of the standard workhorses in the study of networks in areas such as epidemiology [72] and community detection [47]. An important problem for such random graph models is to estimate probabilities of non-typical structural behaviors, particularly when the system size is large. Examples of such behavior include, graph components that are larger or smaller than that predicted by the law of large number analysis or degree distributions within components that deviate significantly from their expected values.

A natural formulation of such problems of rare event probability estimation is through the theory of large deviations. In our work we establish a large deviation principle for the exploration process associated with the configuration model. Proofs rely on a representation of the exploration process as a system of stochastic differential equations driven by Poisson random measures and variational formulas for moments of nonnegative functionals of Poisson random measures. Uniqueness results for certain controlled systems of deterministic equations play a key role in the analysis. Applications of the large deviation results, for studying asymptotic behavior of the degree sequence in large components of the random graphs, are discussed.

(2.iv) Moderate Deviation Principles for Stochastic Differential Equations with Jumps [29]. Large deviation principles for small noise diffusion equations have been extensively studied in the literature. Since the original work of Freidlin and Wentzell(1970), model assumptions have been significantly relaxed and many extensions have been studied in both finite-dimensional and infinite-dimensional settings.

The goal of this work is to study moderate deviation problems for stochastic dynamical systems. In such a study one is concerned with probabilities of deviations of a smaller order than in large deviation theory. Consider for example an independent and identically distributed (iid) sequence $\{Y_i\}_{i\geq 1}$ of $\mathbb{R}^d$-valued zero mean random variables with common probability law ρ. A large deviation principle (LDP) for $S_n = \sum_{i=1}^n Y_i$ will formally say that for $c > 0$

$$\mathbb{P}(|S_n| > nc) \approx \exp\{-n \inf\{I(y) : |y| \geq c\}\},$$

where for $y \in \mathbb{R}^d$, $I(y) = \sup_{\alpha\in\mathbb{R}^d}\{\langle\alpha, y\rangle - \log\int_{\mathbb{R}^d} \exp\langle\alpha, y\rangle\rho(dy)\}$. Now let $\{a_n\}$ be a positive sequence such that $a_n \uparrow \infty$ and $n^{-1/2}a_n \to 0$ as $n \to \infty$ (e.g. $a_n = n^{1/4}$). Then a moderate deviation principle (MDP) for S_n will say that

$$\mathbb{P}(|S_n| > n^{1/2}a_n c) \approx \exp\{-a_n^2 \inf\{I^0(y) : |y| \geq c\}\},$$

where $I^0(y) = \frac{1}{2}\left\langle y, \Sigma^{-1}y\right\rangle$ and $\Sigma = \text{Cov}(Y)$. Thus the moderate deviation principle gives estimates on probabilities of deviations of order $n^{1/2}a_n$ which is of lower order than n and with a rate function that is a quadratic form. Since $a_n \to \infty$ as slowly as desired, moderate deviations bridge the gap between a central limit approximation and a large deviations approximation. Moderate deviation principles for discrete time systems have been extensively studied in Mathematical Statistics (see references in [29]), however for continuous time stochastic dynamical systems such principles are less well studied and none of the prior results consider stochastic dynamical systems with jumps or infinite dimensional models.

In this paper we study moderate deviation principles for finite and infinite dimensional SDE with jumps. For simplicity we consider only settings where the noise is given in terms of a PRM and there is no Brownian component. However, the more general case where both Poisson and Brownian noises are present can be treated similarly. In finite dimensions, the basic stochastic dynamical system we study takes the form

$$X^\varepsilon(t) = x_0 + \int_0^t b(X^\varepsilon(s))ds + \int_{\mathbb{X}\times[0,t]} \varepsilon G(X^\varepsilon(s-), y)N^{\varepsilon^{-1}}(dy, ds).$$

Here $b : \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \times \mathbb{X} \to \mathbb{R}^d$ are suitable coefficients and $N^{\varepsilon^{-1}}$ is a Poisson random measure on $\mathbb{X}_T = \mathbb{X}\times[0, T]$ with intensity measure $\varepsilon^{-1}\nu_T = \varepsilon^{-1}\nu\otimes\lambda_T$, where $\mathbb{X}$ is a locally compact Polish space, ν is a locally finite measure on $\mathbb{X}$, λ_T is the Lebesgue measure on $[0, T]$ and $\varepsilon > 0$ is the scaling parameter. Under conditions X^ε will converge in probability (in a suitable path space) to X^0 given as the solution of the ODE

$$\dot{X}^0(t) = b(X^0(t)) + \int_{\mathbb{X}} G(X^0(t), y)\nu(dy),\ X^0(0) = x_0.$$

The moderate deviations problem for $\{X^\varepsilon\}_{\varepsilon>0}$ corresponds to studying asymptotics of

$$(\varepsilon/a^2(\varepsilon)) \log \mathbb{P}(Y^\varepsilon \in \cdot),$$

where $Y^\varepsilon = (X^\varepsilon - X^0)/a(\varepsilon)$ and $a(\varepsilon) \to 0$, $\varepsilon/a^2(\varepsilon) \to 0$ as $\varepsilon \to 0$. In this paper we establish a moderate deviations principle under suitable conditions on b and G. We in fact give a rather general sufficient condition for a moderate deviation principle to hold for systems driven by Poisson random measures. This sufficient condition covers many finite and infinite dimensional models of interest. A typical infinite dimensional model corresponds to the SPDE

$$dX^\varepsilon(u,t) = (LX^\varepsilon(u,t) + \beta(X^\varepsilon(u,t)))dt + \varepsilon \int_{\mathbb{X}} G(X^\varepsilon(u,t-),u,y)N^{\varepsilon^{-1}}(ds,dy) \qquad (7)$$

$$X^\varepsilon(u,0) = x(u), \quad u \in O \subset \mathbb{R}^d.$$

where L is a suitable differential operator, O is a bounded domain in $\mathbb{R}^d$ and the equation is considered with a suitable boundary condition on ∂O. Here $N^{\varepsilon^{-1}}$ is a PRM as above. The solution of such a SPDE has to be interpreted carefully, since typically solutions for which $LX^\varepsilon(u,t)$ can be defined classically do not exist. We follow the framework of Kallianpur and Xiong(1995), where the solution space is described as the space of RCLL trajectories with values in the dual of a suitable nuclear space. We establish a MDP for such infinite dimensional systems by verifying the sufficient condition given in the work.

3 Stochastic Networks

Problems arising from manufacturing, communication and computer networks lead to processing networks that can be quite large in terms of number of stations, buffers and activities. Furthermore, the network topologies can be quite complex. Adding to the challenge, system managers can exercise dynamic control in form of alternate routing, sequencing of jobs and input control. The goal of a system manager is to minimize some suitable cost function which, for example, could be given in terms of inventory holding cost, throughput times, server idleness times, etc. The objective of this research is to develop techniques for obtaining near optimal control policies for general families of processing networks. Although the networks of interest are too complex to be analyzed directly, one can obtain tractable approximations by appealing to diffusion limit theory. The appeal of this approach is that in the limit lot of extraneous details are washed out and one is able to look at network dynamics in its sharpest relief as a (controlled) diffusion process. Control and stability issues for diffusions are significantly more tractable. Once these issues are resolved one can go back and infer stability properties and good control policies for the underlying network. Some accomplishments of our research are as follows.

(3.i) Construction of Asymptotically Optimal Control for a Stochastic Network from a Free Boundary Problem [31]. An asymptotic framework for optimal control of multiclass stochastic processing networks, using formal diffusion approximations under suitable temporal and spatial scaling, by Brownian control problems (BCP) and their equivalent workload formulations (EWF), has been developed by Harrison (1988). This framework has been implemented in many works for constructing asymptotically optimal control policies for a broad range of stochastic network models. To date all asymptotic optimality results for such networks correspond to settings where the solution of the EWF is a reflected Brownian motion in the positive orthant with normal reflections. In this work we consider a well studied stochastic network which is perhaps the simplest example of a model with more than one dimensional workload process. In the regime considered here, the singular control problem corresponding to the EWF does not have a simple form explicit solution, however by considering an associated free boundary problem one can give a representation for an optimal controlled process as a two dimensional reflected Brownian motion in a Lipschitz domain whose boundary is determined by the solution of the free boundary problem. Using the form of the optimal solution we propose a sequence of control policies, given in terms of suitable thresholds, for the scaled stochastic network control problems and prove that this sequence of policies is asymptotically optimal. As suggested by the solution of the EWF, the policy we propose requires a server to idle under certain conditions which are specified in terms of the thresholds determined from the free boundary.

(3.ii) Control Policies Approaching HGI Performance in Heavy Traffic for Resource Sharing Networks[27].

We consider a general network consisting of I resources (labeled $1, \ldots, I$) with associated capacities C_i, $i = 1, \ldots, I$. Jobs of type $1, \ldots, J$ arrive according to independent Poisson processes with rates depending on the job-type and the job-sizes of different job-type are exponentially distributed with parameters once more depending on the type. Usual assumptions on mutual independence are made. The processing of a job is accomplished by allocating a *flow rate* to it over time and a job departs from the system when the integrated flow rate equals the size of the job. A typical job-type requires simultaneous processing by several resources in the network. This relationship between job-types and resources is described through a $I \times J$ incidence matrix K for which $K_{ij} = 1$ if j-th job-type requires processing by resource i and $K_{ij} = 0$ otherwise. Denoting by $x = (x_1, \ldots, x_J)'$ the vector of flow rates allocated to various job-types at any given time instant, x must satisfy the capcity constraint $Kx \leq C$, where $C = (C_1, \ldots, C_I)'$. Such resource sharing networks have been introduced in the work of Massoulié and Roberts [69] as models for Internet flows.

One of the basic problems for such networks is to construct "good" dynamic control policies that allocate resource capacities to jobs in the system. A "good" performance is usually quantified in terms of an appropriate cost function. One can formulate an optimal stochastic control problem using such a cost function, however in general such control problems are intractable and therefore one considers an asymptotic formulation under a suitable scaling. The paper [57] formulates a Brownian control problem (BCP) that formally approximates the system manager's control under heavy traffic conditions. Since finding optimal solutions of such general Brownian control problems and constructing asymptotically optimal control policies for the network based on such solutions is a notoriously hard problem, the paper [57] proposes a different approach in which the goal is not to seek an asymptotically optimal solution for the network but rather control policies that achieve the so called *Hierarchical Greedy Ideal* (HGI) performance in the heavy traffic limit. Formally speaking, HGI performance is the cost associated with a control in the BCP (which is in general sub-optimal), under which (I) no resource's capacity is underutilized when there is work for that resource in the system, and (II) the total number of jobs of each type at any given instant is the minimum consistent with the vector of workloads for the various resources. Desirability of such control policies has been argued in great detail in [57] through simulation and numerical examples.

Constructing simple form rate allocation policies for broad families of resource sharing networks with associated costs converging to the HGI performance determined from the corresponding BCP, has been a challenging open problem in the field[57]. The goal of this work is to make progress on this open problem. We consider two types of cost criteria, the first is an infinite horizon discounted cost and the second is a long time average cost per unit time. In particular the second cost criterion is analogous to the cost function considered in [57]. We introduce a sequence of rate allocation control policies that are determined in terms of certain thresholds for the scaled queue length processes and prove that, under conditions, the two types of costs associated with these policies converge in the heavy traffic limit to the corresponding HGI performance. This result gives a broad class of resource sharing networks for which the program outlined in [57] can be completed satisfactorily.

4 Some Other Works.

In this section we report on some other works that fall outside the realm of the three main topics discussed above.

(4.i) Source Detection Algorithms for Dynamic Contaminants Based on the Analysis of a Hydrodynamic Limit [1]. In this work we propose and numerically analyze an algorithm for detection of a contaminant source using a dynamic sensor network. The algorithm is motivated using a global probabilistic optimization problem and is based on the analysis of the hydrodynamic limit of a discrete time evolution equation on the lattice under a suitable scaling of time and space. Numerical results illustrating the effectiveness of the algorithm are presented.

(4.ii) Individual confidence intervals for true solutions to stochastic variational inequalities [64]. Stochastic variational inequalities (SVI) provide a means for modeling various optimization and equilibrium problems where data are subject to uncertainty. Often it is necessary to estimate the true SVI solution by the solution of a sample average approximation (SAA) problem. This paper proposes three methods for building confidence intervals for components of the true solution, and those intervals are computable from a single SAA solution. The first two methods use an “indirect approach” that requires initially computing asymptotically exact confidence intervals for the solution to the normal map formulation of the SVI. The third method directly constructs confidence intervals for the true SVI solution; intervals produced with this method meet a minimum specified level of confidence in the same situations for which the first two methods are applicable. We justify the three methods theoretically with weak convergence results, discuss how to implement these methods, and test their performance using two numerical examples.

(4.iii) Critical Random Graphs and the Differential Equations Technique [14]. Over the last few years a wide array of random graph models have been postulated to understand properties of empirically observed networks. Most of these models come with a parameter t (usually related to edge density) and a (model dependent) critical time t_c that specifies when a giant component emerges. There is evidence to support that for a wide class of models, under moment conditions, the nature of this emergence is universal and looks like the classical Erdős-Rényi random graph, in the sense of the critical scaling window and (a) the sizes of the components in this window (all maximal component sizes scaling like $n^{2/3}$) and (b) the structure of components (rescaled by $n^{-1/3}$) converge to random fractals related to the continuum random tree. The aim of this note is to give a non-technical overview of recent breakthroughs in this area, emphasizing a particular tool in proving such results called the differential equations technique first developed and used extensively in probabilistic combinatorics in the work of Wormald(1995, 1999) and developed in the context of critical random graphs by the authors and their collaborators.

(4.iv) On the multi-dimensional skew Brownian motion [4]. We provide a new, concise proof of weak existence and uniqueness of solutions to the stochastic differential equation for the multidimensional skew Brownian motion. We also present an application to Brownian particles with skew-elastic collisions.

Let Σ denote the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$ in $\mathbb{R}^d$, $d \geq 1$, and let a vector field $b : \Sigma \to \mathbb{R}^d$ be given on it. Consider the stochastic differential equation (SDE), for a process X taking values in $\mathbb{R}^d$, of the form

$$X(t) = x + W(t) + \int_0^t b(X(s))dL(s), \qquad t \geq 0, \tag{8}$$

where $x \in \mathbb{R}^d$, W is a standard d-dimensional Brownian motion and L is the local time of X at Σ, and b is bounded and Lipschitz, and satisfies $b_1(x) \in [-1, 1]$ for all $x \in \Sigma$. This paper provides a new proof of weak existence and uniqueness of solutions to (8). A more general equation that allows for a bounded, measurable drift coefficient is also treated. These results are known by the work of Takanobu(1986,1987). Our purpose here is to provide a much shorter proof. Takanobu studies in a sequence of two papers an equation of the form (8) with general drift and diffusion coefficients. The first paper treats existence of weak solutions and the second proves uniqueness. Existence is shown by constructing a suitable approximating diffusion with a non-singular drift while uniqueness relies on rather involved arguments from Brownian excursion theory. While our results are a special case of those established by Takanobu, the proof (of both existence and uniqueness) provided here is much shorter and more elementary. Existence of solutions is established by constructing a 'skew random walk' and showing that in diffusion scaling it converges in distribution to a weak solution of the equation. The proof of weak uniqueness provided here is inspired by the technique used in Weinryb(1984) in a one-dimensional setting with time-varying skewness. As an application we consider a model introduced by Fernholz et al. (2013) for the dynamics of a pair of 1-dimensional Brownian particles X_1 and X_2 that exhibit various possible types of interaction when they collide. The equations involve the local time at zero of the relative position, and the types of interaction are determined by the coefficients in front of the local time terms. In contrast to Fernholz et al. our work allows for state-dependent local time coefficients. This allows one to model variability in the type of collision, where the type is determined by the collision position.

5 Ph.D. Advising

(5.i) Ph.D. Dissertation Abstract of Ruoyu Wu.

Weakly interacting particle systems have been widely used as models in many areas, including, but not limited to, communication systems, mathematical finance, chemical and biological systems, and social sciences. In this dissertation, we establish law of large numbers (LLN), central limit theorems (CLT), large deviation principles (LDP) and moderate deviation principles (MDP) for several types of such systems. The dissertation consists of two parts. In the first part, we are mainly concerned with LLN and CLT. We begin by studying weakly interacting multi-type particle systems that arise in neurosciences to model a network of interacting spiking neurons. We prove a CLT showing the centered and suitably normalized empirical measures converge in distribution to a Gaussian random field. This result can in particular be applied to single-type systems to characterize the joint asymptotic behavior of large disjoint subpopulations. We then establish a CLT for a multi-type model where each particle is affected by a common source of noise. Here the limit is not Gaussian but rather described through a suitable Gaussian mixture. Next, we consider weakly interacting particle systems in a setting where not every pair of particles interacts, but rather particle interactions are governed by Erdős-Rényi random graphs and an interaction between a pair of particles occurs only when there is a corresponding edge in the graph. Edges can form and break down independently as time evolves. We prove a LLN and CLT under conditions on the edge probabilities. The second part of this dissertation concerns MDP and LDP for certain weakly interacting particle systems. We study interacting systems of both diffusions and of pure jump Markov processes with a countable state space. We are interested in estimating probabilities of moderate deviations of empirical measure processes, from the LLN limit. For both systems a MDP is established which is formulated in terms of a LDP with an appropriate speed function, for suitably centered and normalized empirical measure processes. Finally, we study particle approximations for certain nonlinear heat equations using a system of Brownian motions with killing. A LLN and LDP for sub-probability measure valued processes given as the empirical measure of the alive Brownian particles are proved. We also give, as a byproduct, a convenient variational representation for expectations of nonnegative functionals of Brownian motions along with an i.i.d. sequence of random variables.

(5.ii) Ph.D. Dissertation Abstract of Abhishek Pal Majumder.

In probability and statistics limit theorems are some of the fundamental tools that rigorously justify a proposed approximation procedure. However, typically such results fail to explain how good is the approximation. In order to answer such a question in a precise quantitative way one needs to develop the notion of convergence rates in terms of either higher order asymptotics or non-asymptotic bounds. In this dissertation, two different problems are studied with a focus on quantitative convergence rates. In first part, we consider a weakly interacting particle system in discrete time, approximating a nonlinear dynamical system. We deduce a uniform in time concentration bound for the Wasserstein-1 distance of the empirical measure of the particles and the law of the corresponding deterministic nonlinear Markov process that is obtained through taking the particle limit. Many authors have looked at similar formulations but under a restrictive compactness assumption on the particle domain. Here we work in a setting where particles take values in a non-compact domain and study several time asymptotics and large particle limit properties of the system. We establish uniform in time propagation of chaos along with a rate of convergence and also uniform in time concentration estimates. We also study another discrete time system that models active chemotaxis of particles which move preferentially towards higher chemical concentration and themselves release chemicals into the medium dynamically modify the chemical field. Long time behavior of this system is studied. Second part of the dissertation is focused on higher order asymptotics of Generalized Fiducial inference. It is a relevant inferential procedure in standard parametric inference where no prior information of unknown parameter is available in practice. Traditionally in Bayesian paradigm, people propose posterior distribution based on the non-informative priors but imposition of any prior measure on parameter space is contrary to the no-information belief (according to Fishers philosophy). Generalized Fiducial inference is one such remedy in this context where the proposed distribution on the parameter space is only based on the data generating equation. In this part of dissertation we established a higher order expansion of the asymptotic coverage of one-sided Fiducial quantile. We also studied further and found out the space of desired transformations in certain examples, under which the transformed data generating equation yields first order matching Fiducial distribution.

6 Postdoctoral Fellow Mentoring

During 2014-2017, Dr. Louis Fan, Dr. Dane Johnson and Dr. Michael Perlmutter were postdoctoral fellows partially supported by the grant.

In a joint work with the PI, Johnson is studying heavy traffic limits of controlled stochastic networks. The goal is to develop control policies that have provable good asymptotic performance properties. Paper based on this work is expected to be completed in two months.

Perlmutter, jointly with PI and Dr. Sayan Banerjee is studying large deviations from hydrodynamic limits for certain lattice models for diffusions with nearest neighbor interaction. The paper based on this work is close to completion. *Placement.* Dr. Perlmutter will start a three year postdoctoral position at Michigan State University this Fall.

Fan has completed two papers [23], [24]. The first paper studies a system of interacting diffusions that models chemotaxis of biological cells in a chemical field that is dynamically modified through the collective contributions from the particles. This paper has now appeared in *Elec. J. Prob.* A second paper together with PI and former graduate student Wu on large deviation properties of certain Brownian particle systems with killing has now appeared in *J. Theor. Prob.*. *Placement.* Dr. Fan started a three year postdoctoral position at U. Wisconsin at Madison in Fall 2015.

7 Invited Presentations.

Conferences and Workshops.

- *Scaling Limits for Large Stochastic Networks.* The 39th Conference on Stochastic Processes and their Applications, July 24-28, 2017, Moscow, Russia.
- *Moderate Deviation Principles for Weakly Interacting Particle Systems.* The 4th Institute of Mathematical Statistics Asia Pacific Rim Meeting, June 27-30, 2016, The Chinese University of Hong Kong.
- *Control of Large Stochastic Networks With Mean Eld Interaction.* World Congress in Probability and Statistics, July 11–15, 2016, The Fields Institute, Toronto.
- *Large and Moderate Deviation Problems for Weakly Interacting Markov Processes.* The 8th International Conference on Stochastic Analysis and Its Applications, June 13–17, 2016, Beijing, China.
- *Asymptotically optimal control, Brownian control problems, and free boundary problems.* IMA Special Workshop: Reflected Brownian Motions, Stochastic Networks, and their Applications June 25-27, 2015, IMA Minnesota.

Seminars and Colloquiums.

- *Control of large stochastic networks with mean field interaction.* Distinguished Lecture Series, Industrial Engineering and Operations Research Department, University of California at Berkeley, November 21, 2016.
- *Moderate deviation principles for stochastic dynamical systems.* Boston University, Math Colloquium, March 27, 2015.
- *Moderate Deviation Principles for Stochastic Dynamical Systems.* University of Maryland, Probability Seminar, April 6, 2015.
- *Lyapunov functions and local stability for finite state nonlinear Markov processes.* Duke University, Decision Sciences Seminar, April 1, 2015.
- *Moderate Deviation Principles for Stochastic Dynamical Systems.* North Carolina State University, Probability and Random Systems Seminar, April 8, 2015.

8 Honors.

- Member of the Program Committee for the *5th Institute of Mathematical Statistics Asia Pacific Rim Meeting*.
- Permanent member of the Scientific Committee of the *Seminar on Stochastic Processes* (invited to join in 2015).
- Member of the Program Committee for the *4th Institute of Mathematical Statistics Asia Pacific Rim Meeting*.

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